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COMPARISON OF TWO APPROACHES OF FINITE ELEMENT METHODS FOR THE LARGE AMPLITUDE FREE VIBRATION ANALYSIS OF UNSTIFFENED AND STIFFENED PLATES

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ABSTRACT

Comparison of nonlinear frequencies by two different approaches of large amplitude free vibration analysis of unstiffened and stiffened plates has been done. The nonlinear frequencies have been determined on the basis of the finite element method. A stiffened plate element has been developed for this purpose. The comparison has been done by employing the same element in both the approaches. The deviation of the results from theoretical and analytical solutions has been discussed.

Keywords: finite element method, large amplitude, free vibration, plate, stiffener

1. INTRODUCTION

Nonlinear vibration problem concerning plates of various geometry and shape in different forms has received considerable attention in recent years. Most of the approaches are by analytical means [1-2] The large amplitude free vibration problem of plates by finite element method is complex due to the treatment of time function. The time function has not yet been catered for, satisfactorily by any investigator dealing with the finite element method due to its complex nature.

The first attempt to apply the finite element method to this problem is due to Mei [3]. In his formulation he has assumed that the inplane forces are constant within each element. Based on the idea of Mei [3], Rao et al [4] have proposed a simple finite element formulation where they have used linearizing technique for the nonlinear strain-displacement relations.However the foregoing formulations are erroneous due to the following reasons.

The linearizing functions used in the nonlinear strain-displacement relations are kept constant in the variational process involved in the formulation of the equations of motion, which makes the formulation incorrect. Dumir and Bhaskar [5] have explained this aspect. Furthermore, the motion may not be harmonic at any instant of time and the nonlinear stiffness matrix is evaluated on the basis of the maximum value of the amplitude that actually varies from zero to its maximum value.

The first error provides less hardening effect whereas the second one provides more hardening effect, which ultimately compensates to some extent and gives results nearer to the analytical solutions.

Prathap and Varadan [6-7] have studied the large amplitude vibration of beams and plates. Their treatment is based on the satisfaction of equations at the instant of maximum displacement and for comparison purpose they have reinterpreted the approach of Woinowsky-Krieger [18]. The displacement parameters used for the evaluation of nonlinear stiffness matrix are taken at the instant of maximum amplitude.

Recently, Gray et al [12] have presented a method based on the finite element method where the ambiguities of the earlier methods are overcome. The major difficulties are associated with the time function and this is taken care of by a proper linearization of the assumed limit cycle time function. This has been discussed latter in this paper. The formulation of the nonlinear stiffness matrix is also been treated properly.

In the present paper both the approaches of Sharma and Varadan [11] and Gray et al [12] have been followed. Analysis has been carried out with a newly developed stiffened plate element and the results have been compared with theoretical and analytical solutions.

2. PROPOSED METHOD

The governing nonlinear free vibration equation can be written as

$$[K_s]\{\delta\} + [M]\{\ddot{\delta}\} = \{0\} \tag{1}$$

where $[K_s]$ is the secant stiffness matrix which is dependent on the amplitude of the vibration. $[M]$ is the overall mass matrices and $\{\delta\}$ is the displacement parameter.
Eq. (1) can be expressed as:

$$[M]\{\ddot{\delta}\} + \left([K_0] + \frac{1}{2}[N_1] + \frac{1}{3}[N_2]\right)\{\delta\} = \{0\} \tag{2}$$

where $[K_0]$ is the linear stiffness matrix, $[N_1]$ and $[N_2]$ are the nonlinear stiffness matrices. $[N_1]$ is linearly and $[N_2]$ is quadratically dependent upon δ .

Eq.(1) consists of a set of nonlinear differential homogeneous equations. These can be solved by different approaches.

A stiffened plate element has been developed for this purpose. The basic flat plate element is a combination of the DKT (Discrete Kirchhoff Triangle) plate bending element (Stricklin et al [13]) and Allman's plane stress element [14].

Starting point in the formulation of the bending and membrane element is a 6- noded triangle having 12 degrees of freedom each. By using constraint conditions, the degrees of freedom are finally reduced to u, v, w, θ_x, θ_y and θ_z at three corner points of the element. The derivation of the stiffness and mass matrices of the plate element is not given here as it has been already discussed in detail by Samanta and Mukhopadhyay [15-14].

The formulation of the stiffener has been made in such a way that the stiffener can be placed anywhere within the plate element. The common shape function is used for both the stiffener and the plate elements. This not only facilitates expressing the stiffness matrix of the stiffener in terms of the parameters of the plate element but also maintains the compatibility of the stiffener with the plate. The fact that the stiffener can be placed anywhere within the plate element, thus obviating the restriction of mesh gradings has enabled the treatment of stiffened plate problems in a more elegant manner. The stiffener element formulation has been discussed in details by Samanta and Mukhopadhyay [15].

The analysis of the large amplitude vibration problem consists of two components. The major part involves with the nonlinear finite element formulation of the structure and the second part is the solution procedure. As the stiffness matrix of the plate and the stiffener element are derived, the nonlinear stiffness matrix has been evaluated in the usual manner [17]. The secant stiffness matrix $[K_s]$ is dependent on the structural deformation which is a function of time. The methods differ in the treatment of time function. Once the amplitude has been determined on the basis of time function, the secant stiffness matrix can be determined easily. The approaches followed in the present paper have been discussed below.

2.1 Approach 1

This approach is similar to the approach of Sharma and Varadan [11] and can be termed as Method 1 (M1). According to this approach, the frequency has been evaluated at the instant of maximum amplitude (point of reversal of motion) when

$$\{\delta\}_{max} = 1, \qquad \{\dot{\delta}\} = \{0\}, \quad \text{and} \quad \{\ddot{\delta}\} = -\omega_n^2 \{\delta\} \tag{3}$$

where $\{\delta\}$, $\{\dot{\delta}\}$ and $\{\ddot{\delta}\}$ are displacement, velocity and acceleration vectors, and ω_n is the nonlinear frequency.

In this approach the solution is first obtained by solving the linear equations ignoring the $[N_1]$ and $[N_2]$ terms of Eq. (2). The linear mode shapes are the starting vector for the non-linear analysis. By the direct iteration procedure the non-linear characteristic frequencies and mode shapes have been calculated and the ratio of the non-linear fundamental frequency to the linear fundamental frequency is evaluated. For a particular amplitude, solutions obtained from the previous amplitude have been taken as the starting vectors.

2.2 Approach 2

In this approach the time function has been taken care of by a proper linearization of the assumed limit cycle time function [12]. This approach has been termed as Method 2 (M2). Gray et al [12] have treated the complex time part with a simple approximation and ultimately made it a constant quantity. The nonlinear stiffness matrix is dependent on the structural deformation, which is a function of time. They have eliminated the time part by expanding the $\cos\omega\tau$ term (τ is the non-dimensional time) and neglecting higher harmonics. With these approximations, they have obtained a result closer to that of Woinowsky-Krieger [18].

Following the approach of Gray et al [12], the system of equations have been reduced to a system containing the transverse displacement only as explained below.

Separating into membrane and flexural parts, Eq. (2) yields

$$\begin{bmatrix}[M_{ff}] & [0]\\ [0] & [M_{mm}]\end{bmatrix}\begin{Bmatrix}\ddot{\delta}_f\\ \ddot{\delta}_m\end{Bmatrix}+\left[\begin{bmatrix}[K_{0ff}] & [K_{0fm}]\\ [K_{0mf}] & [K_{0mm}]\end{bmatrix}+\begin{bmatrix}[K_{1ff}] & [K_{1fm}]\\ [K_{1mf}] & [K_{1mm}]\end{bmatrix}+\begin{bmatrix}[K_{2ff}] & [K_{2fm}]\\ [K_{2mf}] & [K_{2mm}]\end{bmatrix}\right]\begin{Bmatrix}\delta_f\\ \delta_m\end{Bmatrix}=\{0\} \tag{4}$$

where $\{\delta_f\}$ and $\{\delta_m\}$ are flexure and membrane displacements respectively. The subscripts ff and mm have been used to represent the matrices corresponding to flexure and membrane parts where as fm and mf are for coupling terms.

By neglecting the inplane mass for lower frequencies, Eq. (4) can be partitioned and written as two separate equations. Solving these partitioned equations for $\{\delta_m\}$ leads to the following reduced system of equations in terms of the transverse displacements $\{\delta$

$$[M_{ff}]\{\ddot{\delta}_f\}+\{([K_{0ff}]+[K_{1ff}]+[K_{2ff}])-([K_{0fm}]+[K_{1fm}]+[K_{2fm}])$$

$$([K_{0mm}]+[K_{1mm}]+[K_{2mm}])^{-1}([K_{0mf}]+[K_{1mf}]+[K_{2mf}])\}\{\delta_f\}=\{0\} \tag{5}$$

where the relationship

$$\{\delta_m\}=-([K_{0mm}]+[K_{1mm}]+[K_{2mm}])^{-1}([K_{0mf}]+[K_{1mf}]+[K_{2mf}])\{\delta_f\} \tag{6}$$

has been used in deriving the equation.

The matrix equation as written, is in the configuration space and as such, does not lend itself to standard eigen solution algorithm. Thus, the approach to be adopted transforms the problem from the configuration space to a state space, which results in a more standard form of the eigenvalue problem. The solution of the homogeneous problem is sought in the form of

$$\{\delta_f\}=\bar{c}\ \{\phi\}\,e^{\Omega\tau} \tag{7}$$

where $\{\phi\}$ is the complex eigenvector, $\Omega=(\alpha+i\omega)$ is the complex eigenvalue, and $\bar{c}$ is a nonzero (scalar) constant displacement amplitude and τ is the nondimensional time.

Substituting the Eq. (7) into Eq. (5) gives

$$\bar{c}\left(\Omega[M_{ff}]+[K^{*}]\right)\{\phi\}e^{\Omega\tau}=\{0\} \tag{8}$$

where

$$[K^{*}]=\{([K_{0ff}]+[K_{1ff}]+[K_{2ff}])-([K_{0fm}]+[K_{1fm}]+[K_{2fm}])$$

$$([K_{0mm}]+[K_{1mm}]+[K_{2mm}])^{-1}([K_{0mf}]+[K_{1mf}]+[K_{2mf}])\}$$

By expressing $e^{\Omega\tau}$ as a complex quantity in Euler form and requiring both coefficients of $\sin\omega t$ and $\cos\omega t$ to vanish, Eq.(8) can be written as two equations :

$$\bar{c}\; e^{\alpha\tau}\left(\Omega[M_{ff}]+[K^{*}]\right)\{\phi\}\cos\omega\tau=\{0\} \quad (9)$$

$$i\bar{c}\, e^{\alpha\tau}\left(\Omega[M_{ff}]+[K^{*}]\right)\{\phi\}\sin\omega\tau=\{0\} \quad (10)$$

Since $\bar{c}$ is nonzero and the solution sought is for all times greater than zero, both Eqs (9) and (10) represent the same eigenvalue problem. As is generally the case with most nonlinear problems, numerous methodologies are available to obtain linearized solutions. A significant focus of this study has been centered around linearizing the resulting nonlinear eigenvalue problem of Eq. (9) for synchronous motions. This can be accomplished by linearizing Eq. (5) and employing an iterative solution procedure.

Now neglecting higher harmonics, the assumed time function for displacements can be approximated to a quantity that is independent of time. The homogeneous Eq. (4) can be solved in an iterative manner by the technique proposed by Gray et al [12] which consists of using a linearised updated mode with a nonlinear time function approximation (LUM/NTF).

3. NUMERICAL EXAMPLES

Numerical results for large amplitude free vibration analysis have been presented in this section. For simply supported boundary condition, of all the examples presented, the in-plane boundary conditions are $u = v = 0$ at all four edges.

3.1 Unstiffened square plate

A square plate has been analysed by the proposed method. Both types of boundary conditions, simply supported and clamped at all edges have been used for this purpose. The results obtained are presented along with the existing solutions. A quarter plate has been analysed by 8 × 8 mesh division for both types of boundary conditions. The convergence of the nonlinear frequency ratio (ω_n/ω) for the fundamental mode of simply supported and clamped boundary conditions for amplitude ratio, $c/h = 1$ has been presented in Tables 1 and 2 respectively.

Table 1. Convergence of frequency ratio (ω_n/ω) for the fundamental mode of the simply supported square plate for amplitude ratio $c/h = \mathbf{1.0}$

Mesh division in quarter plate	2 × 2	4 × 4	8 × 8	10 × 10
Method 1	1.53259	1.52867	1.53090	1.53128
Method 2	1.41934	1.41501	1.41593	1.41624

Table 2. Convergence of frequency ratio (ω_n/ω) for the fundamental mode of the clamped square plate for amplitude ratio $c/h = \mathbf{1.0}$

Mesh division in quarter plate	2 × 2	4 × 4	8 × 8	10 × 10
Method 1	1.24988	1.22456	1.21876	1.21801
Method 2	1.19320	1.17317	1.16823	1.16761

The nonlinear frequency ratio (ω_n/ω) for the fundamental mode for the case of simply supported boundary conditions by the present methods has been shown in Table 3 along with the analytical solution (based on a perturbation method) of Chu and Harrman [1].

Table 3. Nonlinear frequency ratio (ω_n/ω) for fundamental mode of simply supported square plate

c/h	0.2	0.4	0.6	0.8	1.0
Chu and Harrmann [1]	1.01947	1.07561	1.16252	1.27339	1.40232
Method 1	1.02618	1.10123	1.21648	1.36248	1.53090
% Deviation	0.658	2.382	4.642	6.997	9.169
Method 2	1.01969	1.07672	1.16592	1.28097	1.41593
% Deviation	0.022	0.103	0.293	0.595	0.971

Percentage deviation of the frequencies of the present methods from the analytical solution has also been given in the same table. It can be seen from Table 3 that the results obtained by the present approach of Method 2 have excellent agreement with the analytical solution of Chu and Herrmann [1]. On the other hand the results obtained by Method 1 of present approach are higher compared to the results of Method 2 and analytical solutions. It is expected, because in the case of Method 1, the displacement at the instant of its maximum amplitude has been used for the evaluation of nonlinear stiffness parameters that introduces an overhardening effect.

Table 4 shows the nonlinear frequency parameter (ω_n/ω) of fundamental mode for clamped square plate. This problem has been solved analytically by Yamaki [2] based on Galerkin method. Here also it is found that the present approach of Method 2 agrees excellently with analytical solution of Yamaki [2] rather than present approach of Method 1. Method 1 solutions are marginally higher than the analytical solutions. the explanation are given earlier.

Table 4. Nonlinear frequency ratio (ω_{n}/ω) for fundamental mode of clamped square plate

c/h	0.2	0.4	0.6	0.8	1.0
Yamaki [2]	1.00847	1.02923	1.06609	1.11358	1.16740
Method 1	1.00975	1.03841	1.08432	1.14525	1.21876
% Deviation	0.001	0.892	1.710	2.844	4.400
Method 2	1.00732	1.02895	1.06393	1.11088	1.16823
% Deviation	0.001	0.027	0.203	0.243	0.071

Tables 5 and 6 present the frequency ratio (ω_{n}/ω) for second mode ($m = 2$, $n = 1$, where m = number of half sine waves along the length of the plate, and n = number of half sine waves along the width of the plate) of a simply supported and clamped square plate respectively. Comparison has been made with Mei and Rogers [19]. A good agreement has been found between the present frequencies of Method 2 and with Mei and Rogers [19], whereas the earlier trend has been followed for the case of Method 1.

Table 5. Nonlinear frequency ratio (ω_{n}/ω) for second mode (m=2, n=1) of simply supported square plate

c/h	0.2	0.4	0.6	0.8	1.0
Mei and Rogers [19]	1.0241	1.0930	1.1984	1.3307	1.4822
Method 1	1.03263	1.12595	1.26925	1.45052	1.65894
% Deviation	0.381	1.477	3.128	7.594	7.190
Method 2	1.02454	1.09552	1.20634	1.34910	1.51624
% Deviation	0.043	0.231	0.663	1.383	2.297

3.2 Simply supported cross stiffened square plate

A simply supported square plate stiffened along both the directions by central eccentric stiffeners as shown in Figure. 1 has been analysed by using 8 × 8 mesh division in quarter plate. Nonlinear frequency ratios (ω_{n}/ω) obtained in the present analysis are compared with those of Sheikh and Mukhopadhyay [20] and have been presented in Table 7. Sheikh and Mukhopadhyay have solved the problem by using finite strip method using the same approach of Gray et al [12]. The present results of Method 2 are well agreed with Sheikh and

Mukhopadhyay. The percentage deviation of Method 2 is within 1.35 %, whereas the percentage deviation of method 1 of the present with the solutions of reterence [20] is more than 4 %.

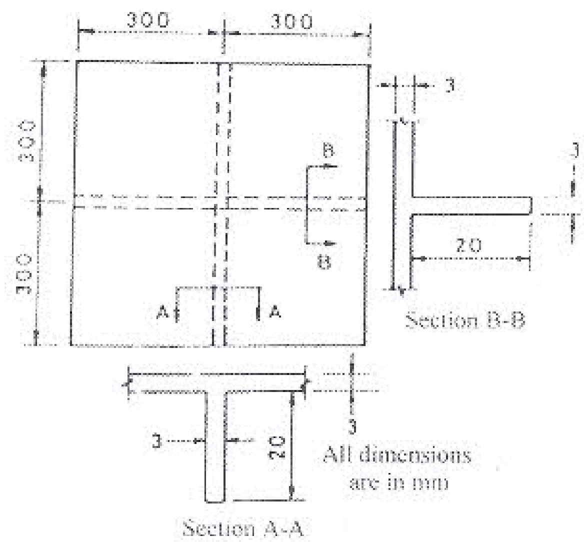

Figure 1 Simply supported cross stiffened square plate

Table 6. Nonlinear frequency ratio (ω_n/ω) for second mode (m=2, n=1) of clamped square plate

c/h	0.2	0.4	0.6	0.8	1.0
Mei and Rogers [19]	1.0152	1.0583	1.1239	1.2064	1.3015
Method 1	1.01907	1.07393	1.15905	1.29801	1.39508
% Deviation	0.381	1.477	3.128	7.594	7.190
Method 2	1.01433	1.05601	1.12163	1.20700	1.30803
% Deviation	0.086	0.216	0.202	0.0497	0.502

Table 7. Nonlinear frequency ratio (ω_n/ω) of a cross stiffened square plate

c/h	0.2	0.4	0.6	0.8	1.0
Sheikh and Mukhopadhyay [20]	1.0048	1.0176	1.0371	1.0615	1.0896
Method 1	1.0093	1.0280	1.0556	1.0912	1.1337
% Deviation	0.448	1.022	1.784	2.798	4.047
Method 2	1.0090	1.0231	1.0440	1.0711	1.1039
% Deviation	0.418	0.541	0.665	0.904	1.312

4. CONCLUSIONS

Comparison of the nonlinear frequencies of unstiffened and stiffened plates has been done between the two approaches, Method 1 (M1) and Method 2 (M2) using a newly developed stiffened plate element. From the tables it has been observed that the results obtained from Approach 2 or Method 2 are closer to the analytical or theoretical solutions. It has also been observed that the results obtained using Method 1 are higher, compared to the analytical solutions. In Method 1 the evaluation of the nonlinear stiffness matrix has been taken at the instant of maximum amplitude, which is incorrect. In this aspect the approach of Method 2 is better than the approach of Method 1. However research is still going on in this field. It can be expected, that the large amplitude problem of unstiffened and stiffened plates will draw the attention of more number of researchers in future as many things can be done in this field by applying the finite element method.

The newly developed stiffened plate element is very effective for the nonlinear analysis of stiffened plates. In future, the present work can be extended to the nonlinear dynamic analysis of stiffened shells which has not yet been reported anywhere to the author's knowledge.

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